Understanding in and about the Mathematics Common Core in the Elementary Grades

Critical Thinking with the Common Core
Middle Tennessee State University
Murfreesboro, TN
June 4, 2013

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Understanding in and about the Mathematics Common Core in the Elementary Grades

1. Historical background
2. The CCSSM process standards
3. General remarks about understanding
4. Understanding in the CCSSM content standards
5. Understanding of problem solving in the CCSSM
6. Additional understandings teachers need
1. Historical background

1989  Curriculum and Evaluation Standards for School Mathematics (NCTM)
      (grade bands K-5, 6-8, 9-12)

2000  Principles and Standards for School Mathematics (NCTM)
      (grade bands preK-2, 3-5, 6-8, 9-12)

2010  Common Core State Standards for Mathematics (NGA-CCSSO)
      (grades K, 1, 2, 3, 4, 5, 6, 7, 8, 9-12)
### 4th grade scores on NAEP 1992-2011

<table>
<thead>
<tr>
<th></th>
<th></th>
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<tbody>
<tr>
<td>Average score</td>
<td>Nation</td>
<td>219</td>
<td>224</td>
<td>240</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>211</td>
<td>220</td>
<td>233</td>
</tr>
</tbody>
</table>

# 4th grade scores on NAEP 1992-2011

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Average score</td>
<td>Nation</td>
<td>219</td>
<td>224</td>
<td>240</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>211</td>
<td>220</td>
<td>233</td>
</tr>
<tr>
<td>% ≥ Basic</td>
<td>Nation</td>
<td>57%</td>
<td>64%</td>
<td>82%</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>47%</td>
<td>59%</td>
<td>75%</td>
</tr>
<tr>
<td>% ≥ Proficient</td>
<td>Nation</td>
<td>17%</td>
<td>22%</td>
<td>40%</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>10%</td>
<td>18%</td>
<td>30%</td>
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</table>

8th grade scores on NAEP 1990-2011

<table>
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<tr>
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<td>Average score</td>
<td>Nation</td>
<td>262</td>
<td>272</td>
<td>283</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>–</td>
<td>262</td>
<td>274</td>
</tr>
<tr>
<td>% ≥ Basic</td>
<td>Nation</td>
<td>51%</td>
<td>62%</td>
<td>72%</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>–</td>
<td>52%</td>
<td>64%</td>
</tr>
<tr>
<td>% ≥ Proficient</td>
<td>Nation</td>
<td>15%</td>
<td>25%</td>
<td>34%</td>
</tr>
<tr>
<td></td>
<td>Tennessee</td>
<td>–</td>
<td>16%</td>
<td>24%</td>
</tr>
</tbody>
</table>

Source: National Center for Education Statistics,
*The Nation’s Report Card, Mathematics 2011*
2. What is in the CCSSM?

CCSSM Table of Contents
(June 2010 pdf online)

Introduction 3
Standards for Mathematical Practice 6
Standards for Mathematical Content 9
Glossary 85
Tables 88
Sample of Works Consulted 91
Comparing numbers of pages of process and content standards

<table>
<thead>
<tr>
<th>Type of standards</th>
<th>NCTM PSSM (2000)</th>
<th>CCSSM (2010)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Process</td>
<td>147</td>
<td>3</td>
</tr>
<tr>
<td>Content</td>
<td>152</td>
<td>76</td>
</tr>
</tbody>
</table>
CCSSM Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
CCSSM
Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.
PSSM
Process Standards Categories

Problem Solving
Reasoning and Proof
Communication
Connections
Representation
## CCSSM Standards for Mathematical Practice and PSSM Processes

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Make sense of problems and persevere in solving them.</td>
<td>Problem Solving</td>
</tr>
<tr>
<td>Reason abstractly and quantitatively.</td>
<td>Reasoning and Proof</td>
</tr>
<tr>
<td>Construct viable arguments and critique the reasoning of others.</td>
<td>Reasoning and Proof, Communication</td>
</tr>
<tr>
<td>Model with mathematics.</td>
<td>(in the PSSM content standards)</td>
</tr>
<tr>
<td>Use appropriate tools strategically.</td>
<td>(in the PSSM content standards)</td>
</tr>
<tr>
<td>(emphasized throughout the CCSSM)</td>
<td>Connections</td>
</tr>
<tr>
<td>(part of understanding in the CCSSM)</td>
<td>Representation</td>
</tr>
</tbody>
</table>
The range of school mathematics in the CCSSM

- Statistics – Yes, but only from grade 6 on
- Metric system – No, only a mention of some units
- Formal logic – No
- Applications of mathematics – Yes, all grades
- Telling time – Yes, grades 2 and 3
- Reading tables – Yes
- Locating places on maps – No
- Sudoku puzzles – No
- Lucky or unlucky numbers – No
- Using a calculator – Mixed message
3. General remarks about understanding in the CCSSM

Understanding mathematics

These Standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as \((a + b)(x + y)\) and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding \((a + b + c)(x + y)\). Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.
Instances of the U-word in the CCSSM

<table>
<thead>
<tr>
<th>Word</th>
<th>Number of instances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understand(s)</td>
<td>147</td>
</tr>
<tr>
<td>Understanding</td>
<td>92</td>
</tr>
<tr>
<td>Understandings</td>
<td>21</td>
</tr>
<tr>
<td>Understood</td>
<td>3</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td><strong>263</strong></td>
</tr>
</tbody>
</table>
Number of instances of the U-word at each grade band of the CCSSM

<table>
<thead>
<tr>
<th>Total</th>
<th>Introduction/Practices</th>
<th>K-5</th>
<th>6-8</th>
<th>9-12</th>
</tr>
</thead>
<tbody>
<tr>
<td>263</td>
<td>25</td>
<td>123</td>
<td>67</td>
<td>47</td>
</tr>
</tbody>
</table>
Early instances of the U-word

“...what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student’s mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from.” (Introduction)
An early instance of the U-word

“…what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student’s mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from.” (Introduction)
Robert Mager’s view of “understanding”

The word “understand” is to be avoided because you cannot observe understanding directly.

*(Preparing Instructional Objectives, 1962, 1984)*
Conflicting views of doing and understanding

“I hear and I forget, I see and I remember, I do and I understand.”
– Confucius (6th century B.C.)

“They can do it but they don’t understand what they are doing”
– teachers (21st century A.D.)
Behaviorists and Cognitivists

Most teachers are both behaviorists and cognitivists. Behaviorists want students to answer questions correctly and do not necessarily care how they got the answer. Cognitivists want to know what students are thinking as they work and typically ask them to show their work.
Another early instance of the U-word

“Mathematical understanding and procedural skill are equally important.” (Introduction)
Is skill a type of understanding?

Richard Skemp (1976) wrote, regarding instrumental (procedural) understanding and relational (conceptual) understanding: “I now believe that there are two effectively different subjects being taught under the same name, ‘mathematics’.” (italics his)
Types of understanding of a mathematical concept

<table>
<thead>
<tr>
<th>Journalist perspective</th>
<th>Mathematics perspective</th>
</tr>
</thead>
<tbody>
<tr>
<td>Understanding how</td>
<td>Skill-algorithm understanding</td>
</tr>
<tr>
<td>Understanding why</td>
<td>Property-proof understanding</td>
</tr>
<tr>
<td>Understanding when</td>
<td>Use-application understanding</td>
</tr>
<tr>
<td>Creating a metaphor</td>
<td>Representation-metaphor understanding</td>
</tr>
</tbody>
</table>
Classifying the instances of the U-word in the CCSSM (#1)

“These Standards define what students should understand and be able to do in their study of mathematics.” (from the introduction)

Classified as:
N: no idea what it means.
Classifying the instances of the U-word in the CCSSM (#2)

“One hallmark of mathematical understanding is the ability to justify why...a particular mathematical statement is true or where a mathematical rule comes from.” (from the Introduction)

Classified as:
P: property-proof understanding
Classifying the instances of the U-word in the CCSSM (#3)

“Understand that in adding two-digit numbers, one adds tens and tens, ones and ones, and sometimes it is necessary to compose a ten.” (1st grade standard)

Classified as:

S: skill-algorithm understanding
Classifying the instances of the U-word in the CCSSM (#4)

“Apply the area and perimeter formulas for rectangles in real world and mathematical problems.” (4th grade standard)

Classified as:
U: use-application understanding
Classifying the instances of the U-word in the CCSSM (#5)

“Understand a fraction as a number on the number line; represent fractions on a number line diagram.”

(3rd grade standard)

Classified as:

R: representation-metaphor understanding
Dimensions of mathematical understanding

Skill-algorithm understanding
from the rote application of an algorithm through the selection and comparison of algorithms to the invention of new algorithms (calculators and computers included)

Property-proof understanding
from the rote justification of a property through the derivation of properties to the proofs of new properties

Use-application understanding
from the rote application of mathematics in the real world through the use of mathematical models to the invention of new models

Representation-metaphor understanding
from the rote representations of mathematical ideas through the analysis of such representations to the invention of new representations
Classifying the instances of the U-word in the CCSSM (#6)

“Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false.” (1st grade standard)

Classified as:
V: vocabulary
Classifying the instances of the U-word in the CCSSM (#7)

“Apply and extend previous understandings of multiplication and division to multiply and divide fractions. Interpret a fraction as division of the numerator by the denominator \( a/b = a \div b \). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.” (5th grade)

Classified as:
P, U, R
Classifying the instances of the U-word in the CCSSM (#8)

“Use place value understanding and properties of operations to add and subtract. Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.”

Classified as: S, P, R
Frequencies of dimensions of understanding represented by the U-words in the CCSSM

<table>
<thead>
<tr>
<th></th>
<th>Totals</th>
<th>Introduction/Practices</th>
<th>K-5</th>
<th>6-8</th>
<th>9-12</th>
</tr>
</thead>
<tbody>
<tr>
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<td>60</td>
<td>1</td>
<td>21</td>
<td>22</td>
<td>16</td>
</tr>
<tr>
<td>Skill-Algorithm</td>
<td>53</td>
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<td>26</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>Property-Proof</td>
<td>177</td>
<td>10</td>
<td>93</td>
<td>39</td>
<td>35</td>
</tr>
<tr>
<td>Use-Application</td>
<td>67</td>
<td>2</td>
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<td>28</td>
<td>9</td>
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<tr>
<td>Representation</td>
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<td>33</td>
<td>29</td>
<td>8</td>
</tr>
<tr>
<td>-Metaphor</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>23</td>
<td>13</td>
<td>3</td>
<td>6</td>
<td>1</td>
</tr>
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<td>0</td>
<td>2</td>
</tr>
<tr>
<td>MAXIMUM POSSIBLE</td>
<td>263</td>
<td>25</td>
<td>123</td>
<td>67</td>
<td>47</td>
</tr>
</tbody>
</table>
Frequencies of dimensions of understanding represented by the U-words in the CCSSM

<table>
<thead>
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<th></th>
<th>Totals</th>
<th>Introduction/Practices</th>
<th>K-5</th>
<th>6-8</th>
<th>9-12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vocabulary</td>
<td>60</td>
<td>1</td>
<td>21</td>
<td>22</td>
<td>16</td>
</tr>
<tr>
<td>Skill-Algorithm</td>
<td>53</td>
<td>2</td>
<td>26</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>Property-Proof</td>
<td>177</td>
<td>10</td>
<td>93</td>
<td>39</td>
<td>35</td>
</tr>
<tr>
<td>Use-Application</td>
<td>67</td>
<td>2</td>
<td>28</td>
<td>28</td>
<td>9</td>
</tr>
<tr>
<td>Representation-Metaphor</td>
<td>70</td>
<td>0</td>
<td>33</td>
<td>29</td>
<td>8</td>
</tr>
<tr>
<td>None (Vague)</td>
<td>23</td>
<td>13</td>
<td>3</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Other</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td><strong>MAXIMUM POSSIBLE</strong></td>
<td><strong>263</strong></td>
<td><strong>25</strong></td>
<td><strong>123</strong></td>
<td><strong>67</strong></td>
<td><strong>47</strong></td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>454</strong></td>
<td><strong>30</strong></td>
<td><strong>203</strong></td>
<td><strong>138</strong></td>
<td><strong>76</strong></td>
</tr>
</tbody>
</table>
“Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.”
(1st grade standard)
Classified as: S, U, R
“Understand addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.”

(1st grade standard)

Classified as: S, U, R
Frequencies of dimensions of understanding represented by the CCSSM non-STEM standards

<table>
<thead>
<tr>
<th></th>
<th>Totals</th>
<th>K-5</th>
<th>6-8</th>
<th>9-12</th>
</tr>
</thead>
<tbody>
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<td>Vocabulary</td>
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<td>18</td>
<td>11</td>
</tr>
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<td>88</td>
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</tr>
<tr>
<td>Property-Proof</td>
<td>175</td>
<td>67</td>
<td>47</td>
<td>61</td>
</tr>
<tr>
<td>Use-Application</td>
<td>134</td>
<td>55</td>
<td>47</td>
<td>32</td>
</tr>
<tr>
<td>Representation-Metaphor</td>
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<td>46</td>
<td>30</td>
<td>32</td>
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<tr>
<td>None (Vague)</td>
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<td>0</td>
</tr>
<tr>
<td>Other</td>
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<td>7</td>
<td>1</td>
<td>2</td>
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<tr>
<td>MAXIMUM POSSIBLE</td>
<td>342</td>
<td>148</td>
<td>81</td>
<td>113</td>
</tr>
<tr>
<td>Totals</td>
<td>342</td>
<td>284</td>
<td>180</td>
<td>176</td>
</tr>
</tbody>
</table>
Robert, age $4 \frac{1}{2}$
Classifying the instances of the U-word in the CCSSM (#10)

“They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.” (Standard for Mathematical Practice 1)

Classified as: PS, problem-solving.
George Polya’s four phases in problem-solving (from *How To Solve It*)

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back
Understanding the problem (from Polya’s *How To Solve It*)

What does the problem say – can it be stated in your own words? What is the unknown? What are the data [givens]? What is the condition [or conditions]? Is it possible to satisfy the condition[s]? Will a figure help? Is it necessary or helpful to introduce suitable notation?
Problems vs. Exercises

A **problem** is a situation that you would like to resolve, and for which you have no algorithm.

An **exercise** is a situation that you would like to resolve, and for which you have an algorithm.
Some Thoughts About Problems

1. Problem ≠ word problem.
2. What is a problem for one person may not be a problem for another.
3. Learning can change a situation from a problem to an exercise.
4. It is not beneficial to tell students how to solve problem after problem. To learn to solve problems, students need to gargle the situation, have fun with it, treat it like a puzzle, solve it on their own.
Heuristics

Draw a picture.
Create a table.
Try simpler numbers.
Try a special case.
Work backwards.
Solve a simpler similar problem.
Solve an equation.
Find the rectangle with the largest area that can be formed by a 30-foot rope.

**Step 1: Understanding the problem.**

Do you know what is a rectangle is, and what is not a rectangle?

What is “area”? How can the area of a rectangle be determined?

What is meant by “forming a rectangle by a 30-foot rope”?
Find the rectangle with the largest area that can be formed by a 30-foot rope.

**Step 2: Applying heuristics to devise a plan.**
Consider a simpler problem. Find the area of any rectangle that can be formed by a 30-foot rope.

*Draw a picture.*
Find the rectangle with the largest area that can be formed by a 30-foot rope.

**Step 2: Applying heuristics to devise a plan.**

Create a table.

<table>
<thead>
<tr>
<th>Length</th>
<th>Width</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 feet</td>
<td>10 feet</td>
<td>50 square feet</td>
</tr>
<tr>
<td>3 feet</td>
<td>12 feet</td>
<td>36 square feet</td>
</tr>
<tr>
<td>6 feet</td>
<td>9 feet</td>
<td>54 square feet</td>
</tr>
<tr>
<td>2 feet</td>
<td>13 feet</td>
<td>26 square feet</td>
</tr>
<tr>
<td>8 feet</td>
<td>7 feet</td>
<td>56 square feet</td>
</tr>
</tbody>
</table>
Find the rectangle with the largest area that can be formed by a 30-foot rope.

**Step 3: Carrying out the plan.**

**Organize the table. (Perhaps add more rows.)**

<table>
<thead>
<tr>
<th>Length</th>
<th>Width</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 feet</td>
<td>13 feet</td>
<td>26 square feet</td>
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<tr>
<td>3 feet</td>
<td>12 feet</td>
<td>36 square feet</td>
</tr>
<tr>
<td>5 feet</td>
<td>10 feet</td>
<td>50 square feet</td>
</tr>
<tr>
<td>6 feet</td>
<td>9 feet</td>
<td>54 square feet</td>
</tr>
<tr>
<td>8 feet</td>
<td>7 feet</td>
<td>56 square feet</td>
</tr>
</tbody>
</table>
Realms of understanding the Common Core from the teacher’s perspective

- Pedagogical content knowledge
- Concept analysis
- Problem analysis
- Connections to other mathematics
PEDAGOGICAL CONTENT KNOWLEDGE

- Designing and preparing for a lesson
- Explaining and representing ideas new to students
- Finding and analyzing student errors
- Responding to questions that learners have about what they are learning
CONCEPT ANALYSIS

• Considering alternate definitions and their consequences (Vocabulary)
• Recognizing and employing alternate algorithms (Skill-algorithm understanding)
• Engaging students in justifying their actions and thinking (Property-proof understanding)
• Dealing with the wide range of applications of the mathematical ideas being taught (Use-application understanding)
• Using representations to clarify concepts (Representation-metaphor understanding)
• Explaining why concepts arose and how they have changed over time (History-culture understanding)
PROBLEM ANALYSIS

• Engaging students in problem solving (understanding the problem)
• Discussing alternate ways of approaching problems with and without calculator and computer technology (devising a plan)
• Examining different student solution methods (carrying out the plan)
• Offering extensions and generalizations of problems (looking back)
CONNECTIONS TO OTHER MATHEMATICS

- Explaining how ideas studied in school relate to ideas students may encounter or have encountered in other mathematics study
- Comparing different resources’ treatments of a mathematical procedure or topic
- Realizing the implications for student learning of spending too little or too much time on a given topic.
CCSSM
Standards for Mathematical Content
Grade 6 includes 29 standards in 10 important content areas.

Ratios and Proportional Relationships
- Understand ratio concepts and use ratio reasoning to solve problems.

The Number System
- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Compute fluently with multi-digit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers.

Expressions and Equations
- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables.

Geometry
- Solve real-world and mathematical problems involving area, surface area, and volume.

Statistics and Probability
- Develop understanding of statistical variability.
- Summarize and describe distributions.
Thank you!

For a copy of the ppt, e-mail z-usiskin@uchicago.edu